Economic Design of Process Mean, Standard Deviation and Screening Limits Based on Burr Distribution

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Abstract: The traditional quality screening problem considers the correlated and normal quality characteristics between product and screening characteristics. However, non-normal characteristics may exist in many practical production and inspection processes. In this paper, the author applies the bivariate Burr's density function in the quality level setting and screening problem to obtain the optimal process mean, standard deviation, and screening limits based on the minimization of the expected total product cost including quality loss for conforming products and rework costs for non-conforming products. A numerical example is given for illustration and a sensitivity analysis is conducted to assess the effects of model parameters on the optimal problem solution.

Keywords: bivariate Burr distribution, screening limits, process mean, standard deviation

Introduction

The traditional quality selection problem includes setting optimal product and process parameters, e.g., process mean, standard deviation, and specification limits. Previous studies on this subject include Springer [1], Kapur and Wang [2], Rahim and Tuffaha [3], and Chen and Lai [4].

Taguchi [5] used the quadratic quality loss function to evaluate product quality. By minimizing bias and variability among product characteristics, product output value is optimized. Taguchi [5] defines product quality as a social loss incurred when the product is shipped to the customer. Attempts to minimize such social loss should account for producer and customer costs resulting from sub-optimal product quality. Taguchi's [5] quadratic quality loss function can connect on-line and off-line quality control methods and can be applied to process control and quality design processes. Many studies have adopted this quality loss function to determine the economic parameters of control chart, variable sampling inspection plans and specification limit settings, e.g., Darwish [6], Jeang [7, 8, 9], Jeang and Lin [10], Darwish and Duffuaa [11], Darwish and Abdulmalek [12], Duffuaa and El-Ga'aly [13, 14], Darwish et al. [15], and Chen et al. [16, 17].

Classical statistical process control (SPC) considers normal quality characteristics and directly measures product characteristics. However, the abnormal quality characteristics may occur in the production process, and such characteristics may not always be directly measured. Hence, the screening characteristic may be used to determine product characteristics.

Burr [18] developed a density function which can represent various probability distributions based on the first four moments of the data. That is, if the mean, standard deviation, skewness coefficient and kurtosis coefficient of the process characteristic can be estimated with reasonably accuracy, then Burr's density function may be applied to fit this data set. Burr's density function has been successfully used to stand for various normal and non-normal probability distributions in SPC research, including economic design of control charts and product tolerance design, e.g., Tsai [19], Chou et al. [20, 21], Chen [22], and Chen and Yeh [23].

Tsai [19] applied the bivariate Burr function to determine a material's optimal tolerance. The present paper modifies Tsai's [19] model to determine a product's optimal process mean and standard deviation and the limits of the screening characteristic. A numerical example is given and a sensitivity analysis is conducted to assess the impact of model parameters on the optimal problem solution.

Literature Review--- Burr Distribution

Burr [18] developed a density function that can cover a wide range of various normal and non-normal distributions. Durling et al. [24] proposed a standard bivariate Burr distribution in which the joint probability distribution of *X* and *Y* obey the bivariate Burr distribution, Burr (*c*, *d*, *p*). From John and Kotz [25], the margin distributions of *Y* and *X* are both Burr distributions with Burr (*c*, *p*) and Burr (*d*, *p*), respectively. The cumulative biavrate Burr distribution function, i.e., *F*(*x*,*y*), the cumulative distribution function of *X*, i.e., *F*(*x*), and the cumulative distribution function of *Y*, i.e., *F*(*y*) are as follows:

$$F(x,y)=1-(1+y^c)^{-p}-(1+x^d)^{-p}+(1+y^c+x^d)^{-p},\ x\geq 0,\ y\geq 0 \quad (1)$$

$$F(x)=1-(1+x^d)^{-p},\ x\geq 0 \quad (2)$$

$$F(y)=1-(1+y^c)^{-p},\ y\geq 0 \quad (3)$$

where $c > 1$, $d>1$, and $p > 1$.

Different combinations of (*c*, *p*) and (*d*, *p*) cover a wide range of the skewness and kurtosis coefficients of various probability density functions, including most of the well-known functions such as normal, Gamma, Beta, and so forth. For example, the normal density function can be approximated by Burr's [18] density function with c = 4.85437 and p = 6.22665. From Johnson and Kotz [25], we have the following results:

$$\mu_x=\frac{\Gamma(1+\frac{1}{d})\Gamma(p-\frac{1}{d})}{\Gamma(p)} \quad (4)$$

$$\mu_y=\frac{\Gamma(1+\frac{1}{c})\Gamma(p-\frac{1}{c})}{\Gamma(p)} \quad (5)$$

$$\sigma_x^2=\frac{\Gamma(1+\frac{2}{d})\Gamma(p-\frac{2}{d})}{\Gamma(p)}-\mu_x^2 \quad (6)$$

$$\sigma_y^2=\frac{\Gamma(1+\frac{2}{c})\Gamma(p-\frac{2}{c})}{\Gamma(p)}-\mu_y^2 \quad (7)$$

Where μ_x is the mean of *X*; μ_y is the mean of *Y*; σ_x is the variance of *X*; σ_y is the variance of *Y*; Γ is called the gamma function, and

$$\Gamma(v)=\int_0^\infty y^{v-1}e^{-y}dy,\ \text{for } v>0.$$

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Modified Tsai's [19] Model

To meet the requirement of quality assurance, assume all products are inspected before being shipped to the customer, using Taguchi's [5] quadratic quality loss function to evaluate product quality. The quality loss function occurs when the screening characteristic is within screening limits. Products are scrapped when the measured value of the screening characteristic is less than the lower screening limit (LSL) or greater than the upper screening limit (USL). Let *X* denote the screening characteristic and *Y* denote the product characteristic. Hence, the expected total product cost, including the quality loss of conforming products and the rework cost of non-conforming products, is

$$\begin{aligned} ETC &= \int_{LSL}^{USL}\int_0^\infty k(y-m_0)^2 f(x,y)dydx + A[\int_0^{LSL} f(x)dx+\int_{USL}^\infty f(x)dx] \\ &= \int_{\mu_x-\delta_1\sigma_x}^{\mu_x+\delta_2\sigma_x}\int_0^\infty k(y-m_0)^2 f(y\mid x)dyf(x)dx + A[\int_0^{\mu_x-\delta_1\sigma_x} f(x)dx+\int_{\mu_x+\delta_2\sigma_x}^\infty f(x)dx] \end{aligned} \quad (8)$$

where k is the quality loss coefficient; m_0 is the target value of product characteristic; A is the scrap cost of product; δ_1 is the lower screening limit coefficient, $\delta_1>0$; δ_2 is the upper screening limit coefficient, $\delta_2>0$;

$$Beta(1+\frac{2}{c},\ p-\frac{2}{c})=\frac{\Gamma(1+\frac{2}{c})\Gamma(p-\frac{2}{c})}{\Gamma(p+1)},\ p-\frac{2}{c}>0 \quad ;$$

$$Beta(1+\frac{1}{c},\ p-\frac{1}{c})=\frac{\Gamma(1+\frac{1}{c})\Gamma(p-\frac{1}{c})}{\Gamma(p+1)},\ p-\frac{1}{c}>0 \quad ;$$

From Tsai [19, p. 4684], Eq. (8) can be rewritten as

$$
\begin{aligned}
ETC &= kpBeta(1+\frac{2}{c},\ p-\frac{2}{c})\{[1+(\mu_x-\delta_1\sigma_x)^d]^{-(p-\frac{2}{c})} \\
&\quad -[1+(\mu_x+\delta_2\sigma_x)^d]^{-(p-\frac{2}{c})}\} \\
&\quad -2km_0\,pBeta(1+\frac{1}{c},\ p-\frac{1}{c})\{[1+(\mu_x-\delta_1\sigma_x)^d]^{-(p-\frac{1}{c})} \\
&\quad -[1+(\mu_x+\delta_2\sigma_x)^d]^{-(p-\frac{1}{c})}\} \\
&\quad +km_0^2\{[1+(\mu_x-\delta_1\sigma_x)^d]^{-p}-[1+(\mu_x+\delta_2\sigma_x)^d]^{-p}\} \\
&\quad +A\{1-[1+(\mu_x-\delta_1\sigma_x)^d]^{-p}+[1+(\mu_x+\delta_2\sigma_x)^d]^{-p}\}
\end{aligned} \qquad (9)
$$

Given that the values of k, m_0, A, d and the parameter p of Burr's density function are known, our objective is to determine the optimal values of c, δ_1 andδ_2 such that the expected total cost of product (*ETC*) is minimized. Based on the optimal values of c, δ_1 and δ_2, the optimal process mean, standard deviation and screening limits of the product/screening characteristics can be consequently determined. Given the difficulty of proving that Eq. (9) is a positively-defined Hessian's matrix with a global minimum value, the pattern search method is applied to obtain an approximate optimal solution.

Numerical Example and Sensitivity Analysis

Consider the case that the probability distribution of the product characteristic can be described by Burr's density function with *p*=4. The probability distribution of the screening characteristic can be described by Burr's density function with *p* = 4 and *d* = 5. The joint probability distribution of the product and screening characteristics obeys the bivariate Burr distribution, Burr (c, 5,4). If the target value m_0=0.6, the quality loss coefficient k =8, the scrap cost A=4, then solving Eq. (9) by the pattern search method leads to the optimal values of c^*=2.63, δ_1^*=1.09, δ_2^*=2.09. Based on these values, the optimal product/screening parameters are process mean μ_y^*=0.449, standard deviation σ_y^*=0.367 lower, screening limit LSL^*=0.034 and upper screening limit USL^*=1.408 with an expected total product *ETC* cost of product of 1.257.

Table 1 and Figs. 1-7 present the sensitivity analysis of the aforementioned numerical example, where the effects of some parameters on the optimal solution of this example are studied. From Table 1 and Figs. 1-7, the following results may be observed:

1. The parameter p of Burr's density function has a major impact on the parameter c of Burr's density function; that is, parameters p and c are mutually correlated.
2. The parameter d of Burr's density function has a major impact on the lower screening limit coefficient (δ_1).
3. The parameter p of Burr's density function has a major impact on the upper screening limit coefficient (δ_2). Meanwhile, the upper screening limit coefficient (δ_2) is significantly influenced by the target value (m_0), quality loss coefficient (k), the scrap cost (A), and the parameter d of Burr's density function.
4. The expected total cost of product is generally affected by the parameter *p* of Burr's density function, target value m_0, and quality loss coefficient (k).
5. The optimal process mean (μ_y) and standard deviation (σ_y) are influenced by the parameter *p* of Burr's density function and the target value (m_0).
6. The parameter d of Burr's density function can clearly impact the optimal value of the lower screening limit (LSL).
7. The target value (m_0), quality loss coefficient (k), scrap cost (A), the parameters p and d of Burr's density function can clearly impact the optimal value of the upper screening limit (USL).

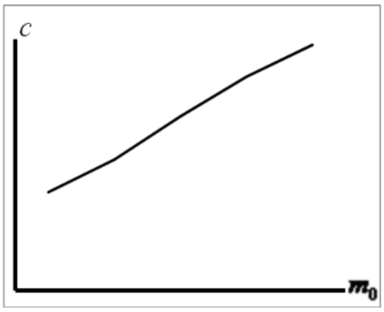

Figure 1. The effect of m_0 on the parameter *c*.

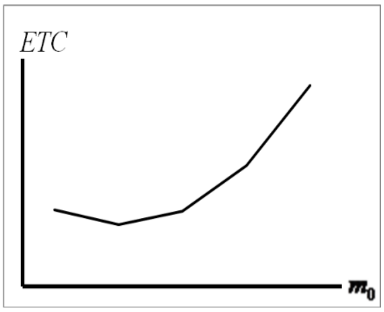

Figure 2. The effect of m_0 on the ETC.

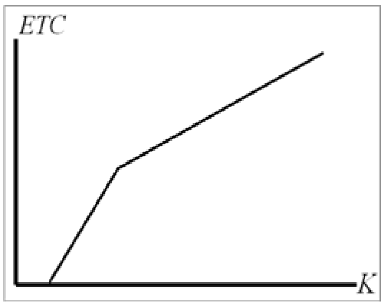

Figure 3. The effect of k on the ETC.

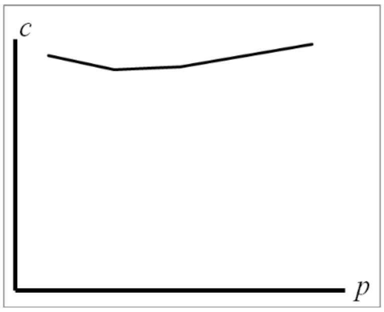

Figure 4. The effect of *p* on the parameter *c*.

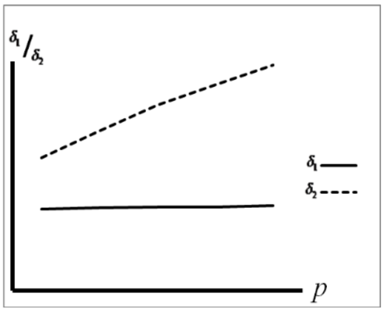

Figure 5. The effect of *p* on the δ_1 and δ_2.

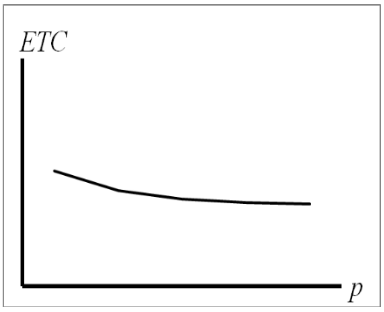

Figure 6. The effect of *p* on the *ETC*.

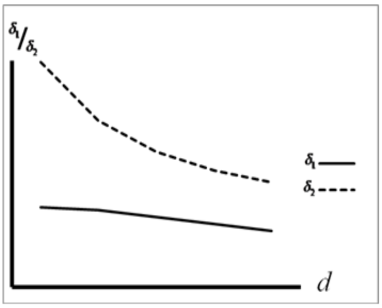

Figure 7. The effect of d on the δ_1 and δ_2.

Table 1. The sensitivity analysis of parameters for numerical example.

m_0	*c*	δ_1	δ_2	*ETC*	μ_y	σ_y	*LSL*	*USL*
1.0	3.71	1.06	2.46	3.353	0.488	0.396	0.047	1.568
0.8	3.23	1.09	2.38	2.015	0.474	0.382	0.034	1.534
0.6	2.63	1.09	2.09	1.257	0.449	0.367	0.034	1.408
0.4	1.98	1.09	1.74	1.027	0.406	0.359	0.034	1.257
0.2	1.48	1.09	1.42	1.271	0.356	0.374	0.034	1.119
K	*c*	δ_1	δ_2	*ETC*	μ_y	σ_y	*LSL*	*USL*
12	2.62	1.09	1.76	1.882	0.449	0.366	0.034	1.266
10	2.62	1.09	1.91	1.570	0.449	0.366	0.034	1.331
8	2.63	1.09	2.09	1.257	0.449	0.367	0.034	1.408
6	2.63	1.09	2.32	0.943	0.449	0.367	0.034	1.508
4	2.63	1.09	3.24	0.029	0.449	0.367	0.034	1.905
A	*c*	δ_1	δ_2	*ETC*	μ_y	σ_y	*LSL*	*USL*
10	2.63	1.09	2.83	1.257	0.449	0.367	0.034	1.728
8	2.63	1.09	2.65	1.257	0.449	0.367	0.034	1.650
6	2.63	1.09	2.41	1.257	0.449	0.367	0.034	1.547
4	2.63	1.09	2.09	1.257	0.449	0.367	0.034	1.408
2	2.60	1.09	1.50	1.251	0.448	0.366	0.034	1.153
p	*c*	δ_1	δ_2	*ETC*	μ_y	σ_y	*LSL*	*USL*
7	2.94	1.11	2.95	1.082	0.379	0.294	0.033	1.567
6	2.80	1.10	2.69	1.101	0.392	0.307	0.036	1.520
5	2.67	1.10	2.43	1.147	0.413	0.329	0.034	1.478
4	2.63	1.09	2.09	1.257	0.449	0.367	0.034	1.408
3	2.80	1.07	1.74	1.517	0.510	0.428	0.035	1.342
d	*c*	δ_1	δ_2	*ETC*	μ_y	σ_y	*LSL*	*USL*
7	2.63	0.87	1.62	1.257	0.449	0.367	0.093	1.277
6	2.62	0.98	1.80	1.257	0.449	0.366	0.061	1.329
5	2.63	1.09	2.09	1.257	0.449	0.367	0.034	1.408
4	2.63	1.19	2.57	1.257	0.449	0.367	0.012	1.533
3	2.63	1.24	3.47	1.257	0.449	0.367	0.0002	1.770

Conclusions

We propose a quality level setting model to determine the optimal process mean, standard deviation and screening limits, given that the probability distribution of product/screening characteristics can be fitted by Burr's density function, which can describe a wide range of various normal and non-normal distributions. We seek to determine the optimal values of Burr's density function (*c*) and the screening limit coefficients (δ_1 and δ_2) to minimize the expected total cost of product (*ETC*), including the quality loss of conforming products and scrap cost of non-conforming products. Based on the optimal values of c, δ_1 and δ_2, the optimal process mean, standard deviation and screening limits of product/screening characteristic can be consequently determined.

A numerical example is given to illustrate the solution, and a sensitivity analysis is provided to assess the effects of model parameters on the optimal solution of the quality level setting problem. Numerical results show that the expected total product cost is significantly influenced by the parameter *p* of Burr's density function, the target value (m_0) of product characteristic, and quality loss coefficient (*k*). The present paper only deals with the case of two correlated product/screening characteristics. However, practical screening processes may suffer from inspection errors, and future work should seek to extend the proposed method to quality level setting problems with inspection errors.

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